VECTOR SPACES · DEFINITION

The ten axioms

Five govern addition, four govern scalar multiplication, and one ties the scalar field to the vectors. Nothing here mentions arrows, coordinates or dimension — that is the point of the list.

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AXIOMS

ADDITION 5

1 Closure

$$\mathbf{u} + \mathbf{v} \in V$$

for all $\mathbf{u}, \mathbf{v} \in V$

Addition may not leave the set. The most frequently violated axiom in practice — a set can look like a vector space and fail here alone, which is what disqualifies most of the non-examples.

2 Commutativity

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

for all $\mathbf{u}, \mathbf{v} \in V$

Order does not matter. Worth noticing this is assumed rather than derived — **matrix multiplication** shows what an operation without it looks like.

3 Associativity

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$

Grouping does not matter, so a sum of many vectors needs no parentheses at all. This is what makes **linear combinations** well defined as written.

4 Zero vector

$$\exists\, \mathbf{0} \in V \,:\, \mathbf{v} + \mathbf{0} = \mathbf{v}$$

one such element, for every $\mathbf{v}$

An additive identity. The axiom asserts existence only — uniqueness is a consequence, proved in two lines by assuming two zeros and adding them to each other.

5 Additive inverse

$$\forall\, \mathbf{v}\ \exists\, -\mathbf{v} : \mathbf{v} + (-\mathbf{v}) = \mathbf{0}$$

one inverse per element

Every vector can be undone. Together with the four above this makes V an abelian group under addition — the scalar axioms are what turn a group into a vector space.

SCALAR MULTIPLICATION 4

6 Closure

$$c\mathbf{v} \in V$$

for all scalars c **and** $\mathbf{v} \in V$

Scaling may not leave the set either. Checking this and closure under addition is the whole test for whether a subset is a **subspace** — the other eight are inherited.

7 Associativity of scalars

$$c(d\mathbf{v}) = (cd)\mathbf{v}$$

for all scalars c, d

Scaling twice equals scaling once by the product. The two multiplications are different operations — cd happens in the field, $c(d\mathbf{v})$ happens in V — and the axiom is the claim that they agree.

8 Distributivity over vectors

$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$$

for all scalars c

Scaling spreads over a sum of vectors. This and the next axiom are exactly what **linearity** demands of a transformation — the definition is these two conditions and nothing more.

9 Distributivity over scalars

$$(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$$

for all scalars c, d

The mirror of the axiom above, spreading over a sum of scalars instead. Both are needed and neither implies the other.

TYING THE TWO TOGETHER 1

10 Multiplicative identity

$$1\mathbf{v} = \mathbf{v}$$

1 **the identity of the scalar field**

The one that looks redundant and is not. Define scaling so that every $c\mathbf{v}$ equals $\mathbf{0}$ and the other nine axioms all hold — this axiom is the only thing ruling that out, and it is what ties V to its field of scalars.

A set satisfying all ten is a vector space whatever its elements are — polynomials, matrices, functions, sequences. Everything proved from these applies to all of them at once, which is the whole return on stating them abstractly.