VECTORS · SUMMARY

Properties at a glance

Three properties belong to a vector on its own; three exist only between a pair. The split matters because the zero vector behaves differently on either side of it.

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PROPERTIES

UNCONDITIONAL 2

i **Magnitude** $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$ UNCONDITIONAL

REQUIRES intrinsic — belongs to one vector

A single non-negative number: the length of the arrow, independent of where it is drawn. It is zero exactly for the zero vector, which is the property positive definiteness of the **dot product** records.

iii **Dimension** $\mathbf{v} \in \mathbb{R}^n \Rightarrow \dim = n$ UNCONDITIONAL

REQUIRES intrinsic — fixed by the space

The component count, decided by the space rather than the vector. No operation on vectors changes it — addition and scaling stay inside $\mathbb{R}^n$, which is what makes it a **vector space** rather than merely a set.

CONDITIONAL 4

ii **Direction** $\hat{\mathbf{v}} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ CONDITIONAL

HOLDS WHEN $\mathbf{v} \neq \mathbf{0}$

FAILS WHEN $\mathbf{v} = \mathbf{0}$ — division by zero

The unit vector pointing the same way. The zero vector has no direction at all, and this is not a convention that could have gone otherwise: the definition divides by a magnitude of zero. Every later statement about $\mathbf{0}$ traces back here.

iv **Equality** $\mathbf{a} = \mathbf{b} \iff a_i = b_i$ for every i CONDITIONAL

HOLDS WHEN both in the same $\mathbb{R}^n$

All-or-nothing: one differing component breaks it. Vectors in different spaces are not unequal so much as incomparable — the question does not arise.

v **Parallelism** $\mathbf{a} \parallel \mathbf{b} \iff \mathbf{a} = c\mathbf{b}$ for some scalar c CONDITIONAL

HOLDS WHEN either vector may be $\mathbf{0}$

COMMON ERROR assuming parallel means same direction — $c < 0$ gives opposite directions

A relation between two vectors, not a property of either. The zero vector is parallel to everything by convention, which is consistent because $\mathbf{0} = 0 \cdot \mathbf{b}$ holds for every $\mathbf{b}$. In $\mathbb{R}^3$ the **cross product** detects this: parallel vectors have $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

vi **Orthogonality** $\mathbf{a} \perp \mathbf{b} \iff \mathbf{a} \cdot \mathbf{b} = 0$ CONDITIONAL

HOLDS WHEN either vector may be $\mathbf{0}$

Also relational. The zero vector is orthogonal to everything, since $\mathbf{0} \cdot \mathbf{b} = 0$ always — so it is simultaneously parallel and orthogonal to every vector, the only one for which both hold. See **orthogonality** for what this makes possible.

> **THE ZERO VECTOR CASE**
>
> 0 · b = 0 for every b → 0 ⊥ b
> 0 = 0 · b for every b → 0 ∥ b

Notice how often the zero vector is the exception. It has no direction, is parallel to everything, and is orthogonal to everything — three conventions that look arbitrary until you see they all follow from $\mathbf{0}$ having magnitude zero and no defined unit vector.