VECTORS · ALGEBRAIC FOUNDATION

The eight laws the operations obey

Four govern addition, four govern scalar multiplication. Together with closure they are exactly the **vector space axioms** — which is why anything proved from them holds for polynomials and matrices too, not just arrows.

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LAWS

UNCONDITIONAL 8

i Commutativity of addition $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

UNCONDITIONAL

REQUIRES component-wise, so it inherits from $\mathbb{R}$

Order of addition does not matter. Geometrically, the parallelogram is the same figure traversed two ways — tip-to-tail in either order lands on the same diagonal.

ii Associativity of addition $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

UNCONDITIONAL

REQUIRES any three vectors in the same $\mathbb{R}^n$

Grouping does not matter, so a chain of additions needs no parentheses. This is what lets **linear combinations** be written as a single sum.

iii Additive identity $\mathbf{a} + \mathbf{0} = \mathbf{a}$

UNCONDITIONAL

REQUIRES $\mathbf{0}$ the vector of all zeros

The zero vector leaves everything unchanged. It is the origin — the one point every **subspace** must contain, since without it there is no identity to add.

iv Additive inverse $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$

UNCONDITIONAL

REQUIRES one inverse per vector

Every vector can be undone by the arrow pointing the opposite way. This is what makes subtraction definable at all — $\mathbf{a} - \mathbf{b}$ is shorthand for $\mathbf{a} + (-\mathbf{b})$, not a separate operation.

v Associativity with scalars $c(d\mathbf{a}) = (cd)\mathbf{a}$

UNCONDITIONAL

REQUIRES any scalars c, d

Scaling twice equals scaling once by the product. Note the two multiplications are different operations — cd happens between numbers, $c(d\mathbf{a})$ between a number and a vector — and the law is the claim that they agree.

vi Distributivity over vector addition $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$

UNCONDITIONAL

REQUIRES any scalar c

Scaling spreads over a sum of vectors. Together with the next law this is precisely what **linearity** demands of a transformation — the definition is these two conditions and nothing more.

vii Distributivity over scalar addition $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$

UNCONDITIONAL

REQUIRES any scalars c, d

The mirror of the law above — spreading over a sum of scalars rather than of vectors. Both are needed, and neither follows from the other.

viii Multiplicative identity $1\mathbf{a} = \mathbf{a}$

UNCONDITIONAL

REQUIRES 1 the scalar identity

COMMON ERROR dismissing this as trivial — without it, defining $c\mathbf{a} = \mathbf{0}$ for every c satisfies all seven laws above

The one that looks like it need not be stated. It is what ties the scalars to the vectors: drop it and scaling could collapse everything to $\mathbf{0}$ while every other law still held.

Every law here is unconditional in $\mathbb{R}^n$, which is what makes the space well behaved and also what makes it unremarkable. The interesting cases are elsewhere: **matrix multiplication** breaks commutativity and cancellation, and the **dot product** breaks closure. Seeing what a space looks like when nothing fails is what makes those failures legible.