TRANSFORMATIONS · SIGNATURES

Recognising a family from its matrix

Each family has an algebraic fingerprint — an identity its matrix satisfies and a spectrum it must have. Given an unfamiliar matrix, these are what identify it without drawing anything.

5
FAMILIES

RIGID — LENGTHS PRESERVED 2

1

Rotation

eigenvalues $e^{\pm i\theta}$ — complex pair

$$R^{T} R = I, \quad \det R = 1$$

Orthogonal with positive determinant. Over $\mathbb{R}^2$ a plane rotation has no eigenvectors at all, since no direction survives unturned — which is exactly what the complex pair records. In $\mathbb{R}^3$ there is one real eigenvalue 1, and its eigenvector is the axis.

2

Reflection

eigenvalues $+1$ on the mirror, -1 across it

$$H^2 = I, \quad H^{T} = H, \quad \det H = -1$$

Involutory and symmetric. Applying it twice returns the original, so H is its own inverse. The eigenvalue -1 is what the negative determinant reports — orientation reverses, and no rotation can do that.

NON-RIGID — LENGTHS CHANGE 2

3

Uniform scaling

eigenvalue c, repeated n times

$$A = cI, \quad \det A = c^n$$

The only family commuting with everything, since cI passes through any product. Every direction is an eigenvector, so the matrix is diagonal in every basis — the extreme opposite of a shear.

4

Shear

eigenvalue 1 repeated — defective

triangular, 1s on the diagonal, $\det = 1$

Area is preserved yet nothing is rigid. The repeated eigenvalue has only one eigenvector rather than two, so a shear cannot be **diagonalised** — the standard example of algebraic multiplicity exceeding geometric.

NOT INVERTIBLE 1

5

Orthogonal projection

eigenvalues 0 and 1 only

$$P^2 = P, \quad P^{T} = P, \quad \det P = 0$$

Idempotent and symmetric. The zero determinant is not incidental — a projection collapses the perpendicular directions and nothing recovers them, so it is the only family here with no inverse. See **projection properties**.

The determinant sorts them at a glance: rotations and shears preserve area, reflections flip it, projections destroy it, and scaling multiplies it by c^n. Two families share $\det = 1$ — rotation and shear — and the eigenvalues separate them: a rotation has a complex pair, a shear has a repeated real one and too few eigenvectors to diagonalise.