MATRIX · TRACE

Trace properties at a glance

Seven identities, each linked to the section that proves it. What holds them together is that the trace sees only the diagonal — every entry below either exploits that or is limited by it.

07
PROPERTIES

UNCONDITIONAL — 3

i — Linearity $\operatorname{tr}(cA + dB) = c\operatorname{tr}(A) + d\operatorname{tr}(B)$

UNCONDITIONAL

REQUIRES A, B both $n \times n$

The trace is a linear functional on the space of $n \times n$ matrices — it maps matrices to scalars while respecting addition and scaling. That is a stronger statement than the formula alone: it means the trace lives in the dual space, which is what makes the **Frobenius inner product** possible.

ii — Transpose invariance $\operatorname{tr}(A^{\mathsf T}) = \operatorname{tr}(A)$

UNCONDITIONAL

REQUIRES A square

Transposition reflects entries across the main diagonal, and the diagonal entries are exactly the ones fixed by that reflection. Since the trace reads only those, it cannot notice.

vi — Commutator trace $\operatorname{tr}(AB - BA) = 0$

UNCONDITIONAL

REQUIRES A, B both $n \times n$

Immediate from the cyclic property and linearity. The consequence is sharper than it looks: since $\operatorname{tr}(I_n) = n \neq 0$, the identity matrix is **never** a commutator. No pair of matrices satisfies $AB - BA = I$ in finite dimensions — which is exactly why the canonical commutation relation of quantum mechanics needs infinite-dimensional operators.

CONDITIONAL — 4

iii — Cyclic property $\operatorname{tr}(AB) = \operatorname{tr}(BA)$

CONDITIONAL

HOLDS WHEN both products defined — A and B need not be square

COMMON ERROR reading it as full commutativity: $\operatorname{tr}(ABC) = \operatorname{tr}(BCA)$ but **not** $\operatorname{tr}(BAC)$

Rotating the factors is allowed; permuting them arbitrarily is not. For three matrices the cyclic rotations $ABC \to BCA \to CAB$ all share a trace, while BAC generally does not. This is the identity everything below is derived from — it is the reason the trace survives a change of basis at all.

> **WHERE IT BREAKS**
>
> `A = [[1, 0], [0, 0]], B = [[0, 1], [0, 0]], C = [[0, 0], [1, 0]]`
> `tr(ABC) = 1 but tr(BAC) = 0`

iv — Similarity invariance $\operatorname{tr}(P^{-1}AP) = \operatorname{tr}(A)$

CONDITIONAL

HOLDS WHEN P **invertible**

A direct consequence of the cyclic property: rotate P^{-1} to the back and it meets P. The consequence is what matters — the trace is a property of the underlying **linear transformation**, not of the basis chosen to write it down. Two matrices with different traces cannot represent the same map.

v — Eigenvalue sum $\operatorname{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

CONDITIONAL

HOLDS WHEN **eigenvalues** counted with algebraic multiplicity, over $\mathbb{C}$

Spectral information read straight off the diagonal, with no characteristic polynomial to solve. Over $\mathbb{R}$ the statement can fail because some eigenvalues may not exist there — a rotation matrix has trace $2\cos\theta$ and no real eigenvalues at all. The sum is still correct once the complex pair is counted.

vii — Frobenius inner product $\langle A, B\rangle_F = \operatorname{tr}(A^{\mathsf T} B)$

CONDITIONAL

HOLDS WHEN real entries; use A^* for the complex case

Gives the space of matrices a geometry — angles, lengths and projections, exactly as the **dot product** does for vectors. Positive definiteness holds because $\langle A, A\rangle_F$ is the sum of every entry squared, so the induced norm is the Frobenius norm.

Two of these are load-bearing for the rest. Linearity makes the trace a functional rather than merely a formula, and the cyclic property is what similarity invariance, the eigenvalue sum and the commutator identity are all derived from.