Dimensions of the standard spaces

Each entry is a space, its dimension, and the basis that produces the count. The split is the point — the same definition gives a number for the first five and no number at all for the last two.

7
SPACES

FINITE DIMENSIONAL — 5

1 **Coordinate space**

basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$

$$\dim(\mathbb{R}^n) = n$$

One standard basis vector per coordinate axis. The count is the number of independent directions, which for $\mathbb{R}^n$ is exactly the number of components — the case every other entry is measured against.

2 **Matrix space**

basis $\{E_{ij}\}$, one per entry position

$$\dim(\mathbb{R}^{m\times n}) = mn$$

Each matrix unit E_{ij} has a single 1 and zeros elsewhere, so there is one basis element per slot. A 3×3 matrix space has dimension 9, not 3 — the shape does not reduce the count.

3 **Polynomials of bounded degree**

basis $\{1, x, x^2, \ldots, x^n\}$

$$\dim(\mathcal{P}_n) = n + 1$$

The constant term counts as a basis direction, so degree ≤ 2 gives dimension 3, not 2. This off-by-one is the most common dimension error, and it comes from reading "degree n" as the count rather than the top index.

4 **Symmetric matrices**

$n \times n$, real, $A = A^T$

$$\dim = \frac{n(n+1)}{2}$$

A **subspace** of $\mathbb{R}^{n\times n}$: the symmetry condition ties each below-diagonal entry to its mirror, so only the diagonal and one triangle are free. Strictly smaller than n^2 for every $n > 1$.

5 **The zero space**

basis is the empty set

$$\dim(\{\mathbf{0}\}) = 0$$

Not dimension one. The zero vector is **linearly dependent** on its own, so it cannot belong to a basis — the empty set spans the zero space vacuously and is independent vacuously.

INFINITE DIMENSIONAL — 2

6 **All polynomials**

no finite basis exists

$$\dim(\mathcal{P}) = \infty$$

For any finite set with maximum degree N, the polynomial x^{N+1} escapes its span — so no finite set can span, and the argument works whatever finite set is proposed.

7 **Continuous functions**

no finite basis exists

$$\dim(C[a, b]) = \infty$$

No finite set of functions generates every continuous function by linear combination. Most of analysis lives here, which is why the finite-dimensional theory has to be stated as such rather than as the general case.

Dimension is a property of the space, not of any basis for it — that is the **basis size theorem**, and it is what makes the counts below well defined rather than an artefact of the basis chosen. The infinite-dimensional entries are not a failure of the definition; they are what the definition says when no finite set spans.