VECTOR SPACES · STANDARD BASES

The default basis of each space

Every space met in practice comes with a natural first choice. The column worth reading is the last one — what a coordinate vector actually contains changes from space to space, even though the definition never does.

4

SPACES

COORDINATE SPACES

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1

Real n-space

$\dim = n$ — coordinates are the components

$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$

Each $\mathbf{e}_i$ has a single 1 in position i. This is the basis where the distinction between a vector and its coordinate vector collapses — they are the same tuple — which is exactly why it hides what coordinates are and why the spaces below are worth stating separately.

FUNCTION AND MATRIX SPACES

2

2

Polynomials of degree ≤ n

$\dim = n + 1$ — **coordinates are the coefficients**

$\{1, x, x^2, \ldots, x^n\}$

The monomial basis. A polynomial is determined by its coefficients, so $a_0 + a_1 x + \cdots + a_n x^n$ has coordinate vector $(a_0, \ldots, a_n)$. Note the count: the constant term is a basis direction, so degree ≤ 2 gives dimension 3.

3

Real m × n matrices

$\dim = mn$ — coordinates are the entries

$\{E_{ij}\}$, one per entry position

Each E_{ij} carries a single 1 at (i, j). Every matrix is a unique combination of them, so the coordinate vector is the matrix read out entry by entry — the shape is a presentation choice, not extra structure.

WHAT MAKES ANY OF THEM WORK

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Unique representation

the c_i are unique

$\mathbf{v} = c_1 \mathbf{b}_1 + \cdots + c_n \mathbf{b}_n$

Independence gives uniqueness and spanning gives existence, so every vector has exactly one coordinate vector. Without both halves the coordinate map is not a function — which is what a basis buys and a mere spanning set does not.

These are standard only in the sense of being the obvious first choice. Many problems are easier in a different basis: an **eigenvector basis** makes matrix powers trivial, an **orthonormal basis** makes projection a dot product, and a Fourier basis turns a signal into frequencies. Choosing the basis is often the whole of solving the problem.