

THE SHARPEST DIAGNOSTIC

When A is $n \times n$, the determinant decides whether the coefficient matrix is adequate; when it is not, the rank comparison with the augmented matrix decides whether the specific $\mathbf{b}$ is reachable.

SQUARE SYSTEM

$$A\mathbf{x} = \mathbf{b}, A \text{ is } n \times n$$

split by $\det(A)$

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CASE 1

$$\det(A) \neq 0$$

Exactly one

solution for every $\mathbf{b}$

the matrix is invertible and the rank is n ; existence and uniqueness both hold universally

$$\mathbf{x} = A^{-1}\mathbf{b}$$

∞ or 0

CASE 2

$$\det(A) = 0$$

Infinitely many or none

depends on $\mathbf{b}$

the matrix is singular and the rank is less than n : infinitely many solutions for $\mathbf{b}$ in the column space, none for $\mathbf{b}$ outside it



TAKEAWAY

With $\det(A) \neq 0$ the right-hand side does not matter. With $\det(A) = 0$ everything depends on whether $\mathbf{b}$ lies in the column space.