DECOMPOSITIONS · SPECTRAL

What the decomposition gives you

Once $A = QDQ^{\top}$ is in hand, every entry below is read off D alone. The orthogonal factor never changes — which is the whole reason functions of the matrix become functions of its eigenvalues.

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READINGS

FUNCTIONS OF THE MATRIX

4

1 Powers

$k \geq 0$ integer; any k if all $\lambda_i > 0$

$$A^k = QD^kQ^{\top}$$

The middle factors collapse: $QDQ^{\top}QDQ^{\top} = QD^2Q^{\top}$, since $Q^{\top}Q = I$. Raising a diagonal matrix to a power means raising each entry, so A^{100} costs no more than A^2 — the reason **diagonalization** is worth the effort at all.

2 Inverse

all $\lambda_i \neq 0$

$$A^{-1} = QD^{-1}Q^{\top}$$

Inverting a diagonal matrix means reciprocating its entries, so the whole inversion costs nothing once the factorization exists. A zero eigenvalue is precisely what makes it fail — the same condition as everywhere else, read off the diagonal.

3 Square root

all $\lambda_i \geq 0$ — positive semi-definite

$$A^{1/2} = QD^{1/2}Q^{\top}$$

A symmetric square root exists exactly when no eigenvalue is negative, and it is unique among positive semi-definite matrices. This is the construction behind **Cholesky** and behind whitening a covariance matrix.

4 Exponential

always defined

$$e^A = Qe^DQ^{\top}$$

No condition at all, since e^x is defined for every real x. This is what solves the linear system of differential equations $\mathbf{x}' = A\mathbf{x}$ — the solution is $e^{At}\mathbf{x}_0$, and the decomposition makes it computable.

SCALARS, STRAIGHT OFF THE DIAGONAL

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5 Trace

always

$$\operatorname{tr}(A) = \lambda_1 + \cdots + \lambda_n$$

The trace is **similarity invariant**, so $\operatorname{tr}(A) = \operatorname{tr}(D)$ and D is diagonal — the sum is immediate.

6 Determinant

always

$$\det(A) = \lambda_1\lambda_2\cdots\lambda_n$$

Same argument with the multiplicative property: $\det Q \cdot \det D \cdot \det Q^{\top} = \det D$, since $\det Q = \pm 1$ and the two cancel.

7 Rank

always

$$\text{count of } \lambda_i \neq 0$$

Multiplying by an invertible matrix preserves rank, so $\operatorname{rank}(A) = \operatorname{rank}(D)$, which is the number of nonzero diagonal entries. For a symmetric matrix, algebraic and geometric multiplicity always agree — so this count is unambiguous in a way it is not for a general matrix.

The pattern is one rule: $f(A) = Qf(D)Q^{\top}$, and $f(D)$ applies f to each diagonal entry. Powers, inverses, roots and exponentials are all the same statement with a different f, which is what makes a symmetric matrix genuinely easier to work with rather than merely nicer to look at.