EIGEN · SUMMARY

Spectral properties at a glance

Nine operations, each linked to the section that derives it. The column worth watching is whether the eigenvectors survive — most operations move the eigenvalues while leaving the directions fixed, and the two exceptions are the informative ones.

09
OPERATIONS

UNCONDITIONAL 4

iv **Powers** A^k has eigenvalues λ^k — UNCONDITIONAL

REQUIRES k a positive integer

Eigenvectors unchanged. Apply A repeatedly to $\mathbf{v}$ and each pass multiplies by λ again. This is why $|\lambda| < 1$ means $A^k \to O$ and $|\lambda| > 1$ means it blows up — the whole of iterative stability sits in this line.

v **Polynomials** $q(A)$ has eigenvalues $q(\lambda_i)$ — UNCONDITIONAL

REQUIRES any polynomial q

Eigenvectors unchanged. Generalises the two entries above — powers and shifts are both special cases. It is also what makes the Cayley–Hamilton theorem believable: the characteristic polynomial sends every eigenvalue to zero.

vi **Shift** $A + cI$ has eigenvalues $\lambda_i + c$ — UNCONDITIONAL

REQUIRES any scalar c

Eigenvectors unchanged. Adding a multiple of the identity slides the whole spectrum along the number line, leaving the directions alone — which is what makes shifted power iteration and the shifted QR algorithm work.

vii **Scalar multiple** cA has eigenvalues $c\lambda_i$ — UNCONDITIONAL

REQUIRES any scalar c

Eigenvectors unchanged. Contrast the shift directly above: scaling stretches the spectrum about zero, shifting translates it. Neither disturbs the eigenvectors.

CONDITIONAL 5

i **Trace** $\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ — CONDITIONAL

HOLDS WHEN eigenvalues counted with algebraic multiplicity, over $\mathbb{C}$

Spectral information read off the diagonal without solving anything. Over $\mathbb{R}$ the sum can appear to fail because some eigenvalues live in $\mathbb{C}$ — a rotation has trace $2\cos\theta$ and no real eigenvalues at all. See **trace properties**.

ii **Determinant** $\det(A) = \lambda_1\lambda_2\cdots\lambda_n$ — CONDITIONAL

HOLDS WHEN with algebraic multiplicity, over $\mathbb{C}$

The product rather than the sum. It gives the invertibility criterion directly: A is **invertible** exactly when no eigenvalue is zero, since a single zero factor kills the whole product.

iii **Inverse** A^{-1} has eigenvalues $1/\lambda_i$ — CONDITIONAL

HOLDS WHEN A invertible — no $\lambda_i = 0$

Eigenvectors unchanged. From $A\mathbf{v} = \lambda_i\mathbf{v}$, multiply both sides by A^{-1}/λ_i and the same $\mathbf{v}$ comes back with the reciprocal. The condition is not a technicality: a zero eigenvalue is precisely what makes the inverse fail to exist.

viii **Transpose** A^T has the same eigenvalues as A — EIGENVECTORS DIFFER

HOLDS WHEN always for the eigenvalues

FAILS WHEN for the eigenvectors — generally different

COMMON ERROR assuming the eigenvectors carry over because the eigenvalues do

Same characteristic polynomial, since $\det(A^T - \lambda I) = \det(A - \lambda I)$. But the eigenvectors of A^T are the **left** eigenvectors of A, a different set. One of only two entries here where the spectrum survives and the directions do not.

> **WITNESS**
> ```
> A = [[1, 1], [0, 2]] — eigenvalues 1, 2
> eigenvector for λ=1: A → (1, 0), Aᵀ → (1, -1)
> ```

ix **Similarity** $P^{-1}AP$ has the same eigenvalues, multiplicities included — EIGENVECTORS MAP

HOLDS WHEN P invertible

Both algebraic and geometric multiplicities are preserved, so the entire spectral picture is basis-independent. The eigenvectors are not lost but relabelled: $\mathbf{v}$ becomes $P^{-1}\mathbf{v}$. That is the whole content of similarity — the same transformation, described from a different basis.

The pattern: anything built from A by polynomials, inversion or shifting keeps the eigenvectors and transforms the eigenvalues by the same rule. Transposition and similarity are the exceptions — they preserve the spectrum but change or relabel the directions, which is exactly why **diagonalization** is a statement about a basis rather than about numbers.