LINEAR SYSTEMS · SOLVABILITY

Every solvability question, answered by rank

Two comparisons settle existence and uniqueness completely. Everything below them is what to do once the answer is known — and the shape of the system determines which of those situations you are in.

6

QUESTIONS

THE TWO QUESTIONS

3

1 Does a solution exist?

equivalently $\mathbf{b} \in \mathrm{Col}(A)$

$$\mathrm{rank}(A) = \mathrm{rank}([A \mid \mathbf{b}])$$

Rouché–Capelli. If the augmented column raises the rank, it contributes a direction the columns of A cannot reach, and the reduction reports this as a row reading $0 = d$. Existence is a question about $\mathbf{b}$, not about A.

2 Is the solution unique?

equivalently $\mathrm{Null}(A) = \{\mathbf{0}\}$

$$\mathrm{rank}(A) = n$$

A pivot in every column means no free variables. Note this says nothing about whether a solution exists — uniqueness is a question about A alone, which is why the two conditions are independent and both must be checked.

3 What does the solution set look like?

when a solution exists

$$\mathbf{x}_p + \mathrm{Null}(A)$$

One particular solution plus the entire null space. The set is a point when the nullity is zero, a line when it is one, a plane when it is two — never anything else, because it is always a translated **subspace**.

WHAT THE SHAPE OF THE SYSTEM IMPLIES

3

4 Square and nonsingular

$m = n$ and $\det A \neq 0$

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Both conditions hold at once and for every $\mathbf{b}$ — the only case where the answer is unconditional. In practice solve by **factorization** rather than by forming the inverse; the formula is a statement about existence, not a method.

5 Overdetermined, inconsistent

$m > n$, **no exact solution**

$$A^{T}A\hat{\mathbf{x}} = A^{T}\mathbf{b}$$

More equations than unknowns usually means no solution at all, so the question changes: least squares minimises $\|A\mathbf{x} - \mathbf{b}\|^2$ instead of solving. The answer is the **projection** of $\mathbf{b}$ onto the column space — the closest reachable point.

6 Underdetermined

$m < n$, **consistent**

infinitely many; choose one

Fewer equations than unknowns leaves free variables, so a solution is never unique and the algebra cannot pick between them. The criterion comes from outside: minimum norm via the pseudoinverse, sparsity, or a physical constraint the model imposes.

The whole of solvability is two numbers compared three ways: $\mathrm{rank}(A)$ against $\mathrm{rank}([A \mid \mathbf{b}])$ decides existence, and $\mathrm{rank}(A)$ against n decides uniqueness. Both come from one **reduction**, which is why the two questions are never asked separately in practice.