LINEAR SYSTEMS · SUMMARY

Everything readable from the RREF

One reduction answers all of these. What differs is where in the reduced form the answer sits — the pivot positions, the free columns, or the augmented column.

8
READINGS

FROM THE PIVOT COUNT — 2

1

Rank

count them; nothing else needed

$\operatorname{rank}(A) =$ number of pivots

Pivots correspond to independent rows and to independent columns simultaneously, which is why **row rank equals column rank** — they are the same pivots counted two ways.

2

Nullity

the number of free columns

$n - \operatorname{rank}(A)$

Every column without a pivot is a free variable, and every free variable contributes one dimension to the null space. Rank-nullity is not a separate theorem here so much as a restatement of the column count.

FROM THE PIVOT POSITIONS — BASES — 3

3

Column space basis

not of the reduced form

pivot columns of the **original** A

Row operations change the column space but preserve which columns are dependent on which. So the reduction tells you *which* columns to take, and you take them from A — the single most common error on this page.

4

Row space basis

of the reduced form, not the original

nonzero rows of the RREF

The exact opposite of the rule above, and the reason both are stated here together. Row operations preserve the row space itself, so the reduced rows still span it — and they are independent, which the original rows may not be.

5

Null space basis

set each free variable to 1 in turn

one vector per free variable

Read from the parametric vector form. The count matches the nullity above, and each basis vector is the solution obtained by turning one free variable on and the rest off.

FROM THE AUGMENTED COLUMN — 3

6

Consistency

$\operatorname{rank}(A) = \operatorname{rank}([A \mid \mathbf{b}])$

no row $[\,0 \cdots 0 \mid d\,]$, $d \neq 0$

Such a row asserts $0 = d$, so the system has no solution. Equivalently $\mathbf{b}$ lies outside the column space — the augmented column raises the rank, and that is what the contradiction row records.

7

Uniqueness

no free variables

a pivot in every column of A

Consistency gives at least one solution, no free variables gives at most one. Both conditions are needed — an inconsistent system has none however many pivots it has.

8

Invertibility

square and full rank

$\operatorname{rref}(A) = I$

The two conditions above at once, for a square matrix: a pivot in every row and every column leaves $_n$ the identity. This is one line of the **invertibility equivalence**, and reduction is how it is checked in practice.

Nothing below requires a second computation. That is the return on reducing: rank, nullity, three bases, consistency, uniqueness and invertibility all come out of a single echelon form, which is why **Gaussian elimination** is the one algorithm worth doing by hand.