MATRIX · RANK

Properties of rank

Rank is constrained rather than computed by these rules — most are inequalities, not identities. That asymmetry is the point: operations can destroy independence but never create it.

08
PROPERTIES

UNCONDITIONAL 3

i Rank bounds $0 \le \operatorname{rank}(A) \le \min(m, n)$

UNCONDITIONAL

REQUIRES A is $m \times n$

Rank cannot exceed either dimension, because it counts independent rows and independent columns simultaneously and those counts agree. A matrix reaching this bound has full rank; anything less is rank-deficient, meaning some row or column is a **linear combination** of the others.

ii Zero matrix $\operatorname{rank}(A) = 0 \iff A = O$

UNCONDITIONAL

The only matrix of rank zero. Any nonzero entry gives at least one independent row, so rank zero and the zero matrix are the same condition stated two ways.

iv Transpose invariance $\operatorname{rank}(A^{\mathsf{T}}) = \operatorname{rank}(A)$

UNCONDITIONAL

REQUIRES any A

A restatement of row rank equalling column rank. It is the reason rank can be defined without saying which of the two is meant, and why a wide matrix and its tall transpose describe the same amount of structure.

CONDITIONAL 5

iii Scalar multiple $\operatorname{rank}(cA) = \operatorname{rank}(A)$

CONDITIONAL

HOLDS WHEN $c \neq 0$

Scaling neither creates nor destroys independence. At $c = 0$ the statement collapses — every rank drops to zero at once — which is why the condition is not decoration.

v Rank of a product $\operatorname{rank}(AB) \le \min(\operatorname{rank}(A), \operatorname{rank}(B))$

UPPER BOUND ONLY

HOLDS WHEN product defined

Multiplication can collapse directions but never create independent ones — the image of AB sits inside the image of A, and its rank is capped by whatever B passes through first. Equality is common but not guaranteed, so this bounds rather than determines.

STRICT INEQUALITY

```
A = [[1, 0], [0, 0]], B = [[0, 0], [0, 1]]
rank(A) = rank(B) = 1, but AB = O with rank 0
```

vi Sylvester's inequality $\operatorname{rank}(A) + \operatorname{rank}(B) - n \le \operatorname{rank}(AB)$

LOWER BOUND

HOLDS WHEN A is $m \times n$ and B is $n \times p$

The companion to the bound above: rank cannot drop arbitrarily far. If both factors have full rank n, the inequality forces $\operatorname{rank}(AB) \ge n$, so the product has full rank too — which is how the invertibility of a product of invertible matrices follows.

vii Rank of a sum $\operatorname{rank}(A + B) \le \operatorname{rank}(A) + \operatorname{rank}(B)$

UPPER BOUND ONLY

HOLDS WHEN A, B same shape

COMMON ERROR reading it as an equality — the sum can have lower rank than either factor

Adding matrices cannot produce more independent directions than the two contribute between them, but it can produce fewer. Unlike the determinant, which has no sum rule at all, rank has a bound — just not an identity.

WITNESS

```
A = I₂, B = -I₂, both rank 2
A + B = O with rank 0, far below either
```

viii Multiplication by an invertible matrix $\operatorname{rank}(PAQ) = \operatorname{rank}(A)$

CONDITIONAL

HOLDS WHEN P and Q **invertible**

An invertible matrix loses nothing, so it cannot change the rank of what it multiplies. This is why **row reduction** computes rank correctly — every elementary operation is multiplication by an invertible matrix, and the echelon form has the same rank as the original.

The one-directional pattern runs through every entry. Multiplication can only lose rank, addition can only be bounded above, and the sole exact identities are the ones where nothing is lost at all — transposition, scaling, and multiplication by an **invertible** matrix.