ORTHOGONALITY · PROJECTIONS

Properties of orthogonal projections

Two conditions define the whole object, and everything below them is a consequence. Idempotence alone gives a projection; symmetry is what makes it the perpendicular one.

07
PROPERTIES

UNCONDITIONAL 4

i Idempotence $P^2 = P$

UNCONDITIONAL

REQUIRES defining condition

Projecting twice changes nothing — once a vector is in W it stays put, so every vector of W is a fixed point of P. This is unusual among **linear transformations**, most of which keep changing a vector on repeated application.

ii Symmetry $P^T = P$

UNCONDITIONAL

REQUIRES defining condition

Self-adjoint with respect to the **dot product**, meaning $P\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot P\mathbf{v}$ for every pair. This is the condition that makes the projection **orthogonal** rather than merely a projection — it forces the residual to be perpendicular to W, not just outside it.

v Complementary projection $I - P$ projects onto $W^\perp$

UNCONDITIONAL

REQUIRES P orthogonal

The residual is itself a projection, onto the orthogonal complement. Check: $(I-P)^2 = I - 2P + P^2 = I - P$ by idempotence, and $(I-P)^T = I - P$ by symmetry — so it satisfies both defining conditions.

vii Eigenvalues $\lambda \in \{0, 1\}$

UNCONDITIONAL

REQUIRES follows from $P^2 = P$

From $P\mathbf{v} = \lambda\mathbf{v}$ and idempotence, $\lambda^2 = \lambda$, so λ is 0 or 1. The eigenvectors for 1 span W, those for 0 span $W^\perp$, and $\operatorname{rank}(P) = \operatorname{tr}(P) = \dim W$ — the trace counts the ones. See **spectral properties**.

CONDITIONAL 2

iv Orthogonal decomposition $\mathbf{b} = P\mathbf{b} + (\mathbf{b} - P\mathbf{b})$

CONDITIONAL

HOLDS WHEN P orthogonal — both conditions

Every vector splits uniquely into a part in W and a part perpendicular to it. Uniqueness is what makes this a decomposition rather than merely one way of writing $\mathbf{b}$, and it is the reason W and $W^\perp$ intersect only at the origin.

vi Minimal distance $\|\mathbf{b} - P\mathbf{b}\| \leq \|\mathbf{b} - \mathbf{w}\|$ for all $\mathbf{w} \in W$

CONDITIONAL

HOLDS WHEN equality iff $\mathbf{w} = P\mathbf{b}$

The projection is the closest point of W to $\mathbf{b}$, and the only one achieving the minimum. This is the entire content of **least squares** — the best approximation is a projection, and the perpendicularity of the residual is why no other point can do better.

FAILS OR UNDEFINED 1

iii Oblique projection $P^2 = P$ but $P^T \neq P$

NOT ORTHOGONAL

FAILS WHEN symmetry is dropped

COMMON ERROR treating idempotence alone as the definition of a projection onto W

Still a projection, and still onto the same subspace — but along a slanted direction rather than the perpendicular one. The two conditions are independent, and this entry exists to show that the second is not implied by the first.

WITNESS

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P = [[1, 1], [0, 0]]
P² = P, so it projects — onto the x-axis
Pᵀ ≠ P, so it projects along the line y = -x, not perpendicular
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Drop symmetry and P still projects onto W, but along some other direction — an oblique projection. The residual is then no longer perpendicular, the distance is no longer minimal, and **least squares** loses its justification entirely. That is how much rests on the second condition.