ONE QUESTION ABOUT THE COLUMNS

The normal equations are always a square $n \times n$ system, and $A^T A$ is always symmetric and positive semi-definite. Whether the least-squares solution is unique depends on the columns of A.

NORMAL EQUATIONS

$$A^T A\hat{\mathbf{x}} = A^T \mathbf{b}$$

split by **whether A has full column rank**

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CASE 1

full column rank

Unique

least-squares solution

$A^T A$ is positive definite and invertible

$$\hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}$$

∞

CASE 2

not full column rank

Infinitely many

solutions

$A^T A$ is singular; the pseudoinverse selects the minimum-norm solution

TAKEAWAY

Every solution produces the same projection $\hat{\mathbf{b}} = A\hat{\mathbf{x}}$; only the $\hat{\mathbf{x}}$ vectors differ.