VECTORS · MAGNITUDE

Properties of the norm

Three axioms define a norm; everything below them is derived. Watch the equality cases — each one identifies a geometric configuration, not just an edge case in the algebra.

06
PROPERTIES

UNCONDITIONAL 3

ii Absolute homogeneity $\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|$

UNCONDITIONAL

REQUIRES any scalar c

COMMON ERROR dropping the absolute value — $\|-2\mathbf{v}\| = 2\|\mathbf{v}\|$, not $-2\|\mathbf{v}\|$

Scaling by c multiplies the length by $|c|$, not by c. The absolute value is doing real work: for $c > 0$ the direction is preserved, for $c < 0$ it reverses, and in both cases the length grows by the same factor. A negative length would violate the axiom above.

iv Reverse triangle inequality $\big|\|\mathbf{u}\| - \|\mathbf{v}\|\big| \leq \|\mathbf{u} - \mathbf{v}\|$

UNCONDITIONAL

REQUIRES derived, not assumed

A lower bound to go with the upper one. It says two vectors of very different lengths cannot be close together — which is what makes the norm a continuous function, and is used constantly in analysis.

v Norm and the dot product $\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$

UNCONDITIONAL

REQUIRES the Euclidean norm specifically

Length is the square root of the self-**dot product**. This is the bridge between the two: positive definiteness of the dot product is the same statement as non-negativity of the norm, seen from the other side.

CONDITIONAL 3

i Non-negativity $\|\mathbf{v}\| \geq 0$

CONDITIONAL

HOLDS WHEN equality iff $\mathbf{v} = \mathbf{0}$

Length is never negative, and only the zero vector has zero length. The second half is the load-bearing part — without it a nonzero vector could have length zero and **normalization** would break, since dividing by the magnitude assumes it is nonzero.

iii Triangle inequality $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

CONDITIONAL

HOLDS WHEN equality iff $\mathbf{u}$ and $\mathbf{v}$ point the same way

The direct path is never longer than the tip-to-tail path. Equality holds exactly when one vector is a non-negative multiple of the other — the triangle degenerates to a line. This is the axiom that makes **distance** a metric, and it follows from the **Cauchy–Schwarz inequality**.

BOTH EXTREMES

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u = (3, 0), v = (4, 0): ‖7‖ = 7 = 3 + 4 equality
u = (3, 0), v = (0, 4): ‖5‖ = 5 < 3 + 4 strict
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vi Unit vector $\|\hat{\mathbf{v}}\| = 1$ where $\hat{\mathbf{v}} = \mathbf{v}/\|\mathbf{v}\|$

CONDITIONAL

HOLDS WHEN $\mathbf{v} \neq \mathbf{0}$

FAILS WHEN $\mathbf{v} = \mathbf{0}$ — division by zero

Immediate from absolute homogeneity with $c = 1/\|\mathbf{v}\|$. The zero vector cannot be normalised, which is the same fact as it having no **direction**.

Any function satisfying the first three is a norm, and the Euclidean length is only one of them. The **taxicab and maximum norms** satisfy all three while measuring distance differently — which is why the axioms are stated separately from the square-root formula that happens to satisfy them.