Which laws survive the product

Matrix multiplication inherits most of scalar arithmetic and breaks four rules that are deeply ingrained from it. Every entry states its condition; every failure exhibits the pair that breaks it.

08 LAWS

UNCONDITIONAL — 5

i — Associativity $(AB)C = A(BC)$ — UNCONDITIONAL

REQUIRES all products defined

Lets you drop parentheses entirely in a chain of products — the reason **matrix powers** are well defined without bracketing.

ii — Left distribution $A(B + C) = AB + AC$ — UNCONDITIONAL

REQUIRES B, C same shape

iii — Right distribution $(A + B)C = AC + BC$ — UNCONDITIONAL

REQUIRES A, B same shape

Left and right distribution are separate statements and each must be checked, precisely because the product does not commute.

iv — Scalar passage $c(AB) = (cA)B = A(cB)$ — UNCONDITIONAL

REQUIRES any scalar c

v — Identity $AI = IA = A$ — UNCONDITIONAL

REQUIRES conformable dimensions

The one place commutativity is guaranteed — I commutes with everything its shape allows.

FAILS OR UNDEFINED — 3

vi — Commutativity $AB \neq BA$ in general — FAILS

HOLDS WHEN both diagonal, or $B = A^{-1}$, or $B = I$

The most consequential failure in the subject. It is why the transpose reverses order and why **similarity** rather than equality is the working notion of "same transformation".

WITNESS

```
A = [[1, 2], [0, 0]], B = [[0, 0], [3, 4]]
AB = [[6, 8], [0, 0]]
BA = [[0, 0], [3, 6]]
```

vii — Zero-product law $AB = O \not\Rightarrow A = O$ or $B = O$ — FAILS

HOLDS WHEN either factor invertible

Matrices are not an integral domain. Two nonzero matrices can annihilate each other — which is what a nontrivial **null space** means at the level of the whole product.

WITNESS

```
A = [[1, 2], [2, 4]], B = [[2, -4], [-1, 2]]
AB = 0, with neither factor zero
```

viii — Cancellation $AB = AC \not\Rightarrow B = C$ — FAILS

HOLDS WHEN A **invertible**

FAILS WHEN A singular

WITNESS

```
A = [[1, 0], [0, 0]]
B = [[1, 1], [1, 1]], C = [[1, 1], [2, 2]]
AB = AC = [[1, 1], [0, 0]] but B ≠ C
```

Four of eight fail. Each failure traces back to the same root — a matrix can collapse directions, and a collapsed direction cannot be recovered. That is what **invertibility** restores, and why every "holds when" below names it.