TRANSFORMATIONS · SUMMARY

How linear maps combine

Every operation here produces another linear map — that closure is what lets the set of all linear maps be a vector space in its own right, rather than merely a collection.

06
OPERATIONS

UNCONDITIONAL — 2

iv Scalar multiple $(cT)(\mathbf{v}) = c \cdot T(\mathbf{v})$

UNCONDITIONAL

REQUIRES any scalar c

Scaling a linear map keeps it linear, including at $c = 0$, which gives the zero map — the additive identity of $\mathrm{L}(V, W)$. Together with the sum above, this is the second of the two operations a vector space needs.

vi The zero vector is fixed $T(\mathbf{0}) = \mathbf{0}$

UNCONDITIONAL

REQUIRES T linear

COMMON ERROR treating this as a definition — it is a consequence, and its failure is a fast disproof

Follows from $T(0 \cdot \mathbf{v}) = 0 \cdot T(\mathbf{v})$. Useful in the contrapositive: any map sending $\mathbf{0}$ somewhere else is not linear, no further work needed. This is why an affine map $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$ with $\mathbf{b} \neq \mathbf{0}$ fails immediately.

> **FAST DISPROOF**
> ```
> T(x, y) = (x + 1, y)
> T(0, 0) = (1, 0) ≠ (0, 0) → not linear
> ```

CONDITIONAL — 4

i Composition $(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$

STAYS LINEAR

HOLDS WHEN the codomain of T matches the domain of S

Apply T's linearity inside S's and the result falls out. In matrix terms this is exactly **matrix multiplication** — which is why that product is defined the way it is, and why it fails to commute: $S \circ T$ and $T \circ S$ are different maps.

ii Inverse $T^{-1} \circ T = I, \quad T \circ T^{-1} = I$

STAYS LINEAR

HOLDS WHEN T is a bijection

FAILS WHEN T has nontrivial kernel or misses part of the codomain

The inverse of a linear bijection is itself linear — not obvious, but true. An invertible map between spaces of equal dimension is an isomorphism, meaning the two spaces are structurally the same object written twice. See the **invertibility equivalence** for the matrix form of the condition.

iii Sum $(S + T)(\mathbf{v}) = S(\mathbf{v}) + T(\mathbf{v})$

STAYS LINEAR

HOLDS WHEN S and T share domain and codomain

Defined pointwise, and linear because both summands are. This is what supplies the addition on $\mathrm{L}(V, W)$ — the operation exists because linear maps are closed under it, not by stipulation.

v The space of linear maps $\dim \mathrm{L}(V, W) = \dim(V) \cdot \dim(W)$

CONDITIONAL

HOLDS WHEN both V and W finite-dimensional

Closure under sum and scalar multiple makes $\mathrm{L}(V, W)$ a **vector space** whose elements happen to be functions. Its dimension is the product of the two, which is the same count as the entries of an $m \times n$ matrix — the **matrix representation** is a basis for it.

Read downward and the entries build on each other: composition and inversion give $\mathrm{L}(V, V)$ the structure of an algebra, while sum and scalar multiple give $\mathrm{L}(V, W)$ the structure of a vector space. The last entry is the two facts combined.