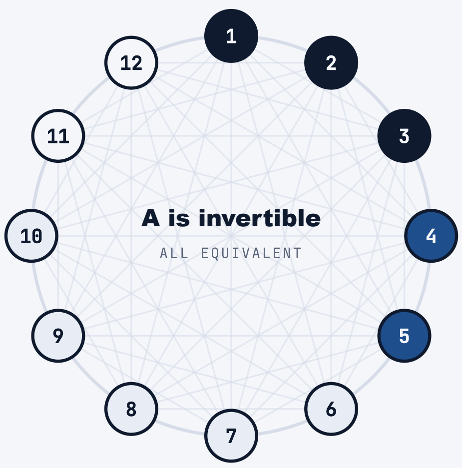
SQUARE MATRICES · TFAE

The invertibility equivalence

Twelve statements about an $n \times n$ matrix, no one of them primary. Any may be taken as the definition and the rest derived — which is why the picture is a ring and not a list.

12 STATEMENTS



1	$\det(A) \neq 0$	DET
2	A is a product of **elementary matrices**	DET
3	0 is not an **eigenvalue** of A	DET
4	$\operatorname{rank}(A) = n$ — full rank	RANK
5	$\operatorname{rref}(A) = I$	RANK
6	The columns of A are **linearly independent**	SPACE
7	The columns of A **span** $\mathbb{R}^n$	SPACE
8	The columns of A form a **basis** for $\mathbb{R}^n$	SPACE
9	The rows of A are linearly independent	SPACE
10	$\operatorname{null}(A) = \{\mathbf{0}\}$	SPACE
11	$A\mathbf{x} = \mathbf{0}$ has only the trivial solution	SYSTEM
12	$A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^n$	SYSTEM

WHAT THE TAGGING EXPOSES

The twelve are not twelve independent claims so much as one fact seen from four sides. $\operatorname{rank}(A) = n$ and $\operatorname{rref}(A) = I$ are the same statement in two notations, and the row condition follows from the column condition because **row rank equals column rank**. What the tagging exposes is how much of the theorem lives in the subspace corner — five of twelve — against a determinant corner that gets stated first and remembered longest.

■ det ×3 ■ rank ×2 □ space ×5 □ system ×2