MATRIX · INVERSE

Properties of the inverse

Seven identities that mirror the rules for reciprocals — with two that do not. The order reversal and the missing sum rule are where the analogy with real numbers stops.

07
IDENTITIES

CONDITIONAL 6

i

Involution $(A^{-1})^{-1} = A$

CONDITIONAL

HOLDS WHEN A invertible

Inverting twice returns the original. The inverse relation is symmetric — if B is the inverse of A then A is the inverse of B, so neither matrix in the pair is privileged.

ii

Inverse of a product $(AB)^{-1} = B^{-1}A^{-1}$

CONDITIONAL

HOLDS WHEN both A and B invertible

COMMON ERROR writing $(AB)^{-1} = A^{-1}B^{-1}$ — the order must reverse

The order reverses, because $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I$ only in that arrangement. This extends to any number of factors: $(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} \cdots A_2^{-1} A_1^{-1}$. The reversal is the same one the **transpose** performs, and both trace back to matrix multiplication not commuting.

iii

Inverse of a transpose $(A^{T})^{-1} = (A^{-1})^{T}$

CONDITIONAL

HOLDS WHEN A invertible

The two operations commute — transpose then invert, or invert then transpose, and the result is the same. Worth contrasting with the product rule directly above: there the order matters, here it does not.

iv

Scalar multiple $(cA)^{-1} = \frac{1}{c} A^{-1}$

CONDITIONAL

HOLDS WHEN A invertible and $c \neq 0$

Scalars pass through and invert. The condition $c \neq 0$ is doing real work — $0 \cdot A$ is the zero matrix, which is never invertible whatever A was.

v

Powers $(A^{k})^{-1} = (A^{-1})^{k}$

CONDITIONAL

HOLDS WHEN A invertible, k a positive integer

Follows from the product rule applied k times; the order reversal cancels out because every factor is the same matrix. This is what lets negative exponents be defined at all, with A^{-k} meaning either side of the identity.

vi

Determinant of the inverse $\det(A^{-1}) = \frac{1}{\det(A)}$

CONDITIONAL

HOLDS WHEN $\det(A) \neq 0$

Immediate from $\det(AB) = \det(A)\det(B)$ applied to $AA^{-1} = I$. It also gives a one-line proof that a singular matrix has no inverse: there is no real number whose product with zero is one. See **determinant properties**.

FAILS OR UNDEFINED 1

vii

Sum and difference $(A+B)^{-1} \neq A^{-1} + B^{-1}$

FAILS

FAILS WHEN in general — no rule exists

There is no product-style identity for the inverse of a sum, and $A+B$ may not even be invertible when both A and B are. The Woodbury identity handles a restricted case, but nothing does in general. This entry exists so the absence is stated rather than inferred from silence.

> **WITNESS**
>
> ```
> A = I, B = -I, both invertible
> A + B = 0, which has no inverse at all
> ```

Every identity here presupposes the matrices involved are invertible, which is exactly what the **equivalence above** characterises. Sum and difference are absent from this list because no rule exists — that absence is the entry, not an omission.