The three axioms

The **dot product** is one inner product among many. These three conditions are what any candidate must satisfy — and each one is responsible for a specific piece of geometry that would otherwise not exist.

06
AXIOMS

UNCONDITIONAL 1

ii

Linearity in the first argument $\langle c\mathbf{u} + d\mathbf{w}, \mathbf{v}\rangle = c\langle \mathbf{u}, \mathbf{v}\rangle + d\langle \mathbf{w}, \mathbf{v}\rangle$

UNCONDITIONAL

REQUIRES any scalars c, d

Combined with symmetry this gives linearity in both arguments — the form is bilinear, so an inner product distributes across **linear combinations** on either side. Only one argument needs stating; symmetry supplies the other.

CONDITIONAL 5

i

Symmetry $\langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{u}\rangle$

CONDITIONAL

HOLDS WHEN real scalars

FAILS WHEN complex scalars — becomes $\langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{u}\rangle$

Order of the arguments is irrelevant over $\mathbb{R}$. Over $\mathbb{C}$ the axiom weakens to conjugate symmetry, and it has to — without the conjugate, positive definiteness below would fail, since $\langle i\mathbf{v}, i\mathbf{v}\rangle$ would come out negative.

iii

Positive definiteness $\langle \mathbf{v}, \mathbf{v}\rangle \geq 0$

CONDITIONAL

HOLDS WHEN equality iff $\mathbf{v} = \mathbf{0}$

The axiom that supplies length. It makes $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}$ real, non-negative, and zero only at the origin — see **norm properties**. Drop the strictness and nonzero vectors could have zero length, which breaks normalization and every distance argument built on it.

iv

Cauchy–Schwarz $|\langle \mathbf{u}, \mathbf{v}\rangle| \leq \|\mathbf{u}\|\,\|\mathbf{v}\|$

DERIVED

HOLDS WHEN equality iff $\mathbf{u}$ and $\mathbf{v}$ are parallel

Not an axiom — a consequence of the three above. It is what makes **angle** definable at all: without it, $\langle \mathbf{u}, \mathbf{v}\rangle/(\|\mathbf{u}\|\|\mathbf{v}\|)$ could fall outside $[-1, 1]$ and have no arccosine.

v

Triangle inequality $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

DERIVED

HOLDS WHEN equality iff $\mathbf{u}, \mathbf{v}$ point the same way

Follows from Cauchy–Schwarz by squaring both sides. This is what makes the induced norm a genuine norm and the induced distance a genuine metric — the three axioms give geometry, but only through this chain.

vi

Pythagorean theorem $\mathbf{u} \perp \mathbf{v} \Rightarrow \|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$

DERIVED

HOLDS WHEN $\langle \mathbf{u}, \mathbf{v}\rangle = 0$

Expand $\|\mathbf{u} + \mathbf{v}\|^2 = \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v}\rangle$ by bilinearity and the cross terms vanish. The classical theorem in $\mathbb{R}^2$ turns out to be a statement about any inner product space — it holds for functions and matrices just as well.

Every geometric notion on this page is derived from these three and nothing else. That is why the same machinery works on function spaces, matrix spaces and polynomial spaces — **Gram–Schmidt** and **projection** never look at what the vectors are, only at the axioms they obey.