VECTOR SPACES · INDEPENDENCE

What operations do to an independent set

Independence is fragile in one direction and robust in the other: removing vectors can never break it, adding them usually does. The two entries that read "may or may not" are the ones worth reading closely.

06
OPERATIONS

UNCONDITIONAL 1

i **Any subset** S independent $\Rightarrow$ every $T \subseteq S$ independent UNCONDITIONAL

REQUIRES any subset, including the empty one

Removing vectors can never create a dependence. A nontrivial combination among the survivors would have been a nontrivial combination in the original set, with zero coefficients on the removed vectors — so the original was never independent to begin with.

CONDITIONAL 3

ii **Adding a vector outside the span** $\mathbf{v} \notin \operatorname{span}(S) \Rightarrow S \cup \{\mathbf{v}\}$ independent CONDITIONAL

HOLDS WHEN $\mathbf{v}$ lies outside $\operatorname{span}(S)$

A genuinely new direction extends the set safely. This is the step that builds a **basis** one vector at a time — keep adding from outside the current span until the span is everything.

v **Reaching the dimension** $|S| = n$ and S independent $\Rightarrow S$ is a basis CONDITIONAL

HOLDS WHEN the space has dimension exactly n

Spanning comes free at the maximum independent size — no separate check needed. The converse shortcut also holds: n vectors that span are automatically independent. Either property plus the right count gives the other.

vi **Removing from a dependent set** dependence may or may not be repaired DEPENDS WHICH

HOLDS WHEN the removed vector participates in every dependence relation

FAILS WHEN other relations survive without it

COMMON ERROR assuming any removal restores independence

Removing a vector that appears in the only relation repairs the set; removing one that does not leaves the relation intact. Which vectors are candidates is read off the **dependence relation** itself, not guessed.

BOTH OUTCOMES

```
S = {(1,0), (2,0), (0,1)} — dependent, one relation
remove (2,0) → independent
remove (0,1) → still dependent: (2,0) = 2(1,0)
```

FAILS OR UNDEFINED 2

iii **Adding a vector inside the span** $\mathbf{v} \in \operatorname{span}(S) \Rightarrow S \cup \{\mathbf{v}\}$ dependent FAILS

FAILS WHEN $\mathbf{v}$ is any combination of S

The new vector is already expressible, and writing it as a combination and moving everything to one side gives a nontrivial relation immediately. Note this includes $\mathbf{v} = \mathbf{0}$, which lies in every span — which is why no independent set can contain the zero vector.

iv **Exceeding the dimension** $|S| > n$ in an n-dimensional space $\Rightarrow$ dependent FAILS

FAILS WHEN more vectors than dimensions

No counting or computation required — the conclusion follows from the size alone. This is the fastest of all the **independence tests**, and the reason a homogeneous system with more unknowns than equations always has a nontrivial solution.

The asymmetry has a single source. Independence is a statement that no nontrivial combination gives $\mathbf{0}$, so a smaller set has fewer combinations to worry about and cannot acquire a relation it did not have. Adding a vector adds combinations, and only one of them needs to vanish for the property to be lost.