LINEAR SYSTEMS · HOMOGENEOUS

Where $A\mathbf{x} = 0$ turns up

Six questions from six different chapters, all of which reduce to solving one homogeneous system. What differs is only what the null space is being called and what its being trivial is taken to mean.

6
SETTINGS

THE SYSTEM ITSELF

2

1 Solution set

always consistent — $\mathbf{x} = \mathbf{0}$ works

$$A\mathbf{x} = \mathbf{0}$$

The solution set is $\mathrm{Null}(A)$, a **subspace** rather than merely a set: closed under addition and scaling, which is the **superposition principle**. One basis vector per free variable, so the dimension is $n - \mathrm{rank}(A)$.

2 Non-homogeneous solution set

when $A\mathbf{x} = \mathbf{b}$ is consistent

$$\mathbf{x} = \mathbf{x}_p + \mathrm{Null}(A)$$

The homogeneous solution supplies the entire shape and size of the general solution; the particular solution only translates it off the origin. This is why the two systems are never solved independently — the same reduction gives both.

THE SAME EQUATION, ASKED DIFFERENTLY

4

3 Linear independence

independent $\iff$ only $\mathbf{c} = \mathbf{0}$

$A\mathbf{c} = \mathbf{0}$, columns $= \mathbf{v}_i$

Independence is the statement that this system has only the trivial solution. When it does not, the nontrivial $\mathbf{c}$ is not merely evidence of dependence — its entries are the coefficients of an explicit **dependence relation** among the vectors.

4 Column dependence relations

one relation per null space basis vector

nontrivial $\mathbf{c}$ with $A\mathbf{c} = \mathbf{0}$

Read the same solution the other way round: each entry of $\mathbf{c}$ says how much of the corresponding column enters the relation, so one column is written as a combination of the others. Which columns are redundant is exactly which are non-pivot.

5 Eigenvectors

λ already known

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

The eigenspace for λ is $\mathrm{Null}(A - \lambda I)$, so finding **eigenvectors** is solving a homogeneous system. A nontrivial solution must exist, which is precisely why $\det(A - \lambda I) = 0$ is the condition defining λ in the first place.

6 Kernel of a transformation

$\ker(T) = \mathrm{Null}(A)$ for the matrix of T

$$T(\mathbf{x}) = \mathbf{0}$$

The abstract statement of the same thing. T is injective exactly when the kernel is trivial — so the size of the null space measures how far a **transformation** is from being one-to-one.

One computation serves all six: reduce, identify the free variables, read off a basis. That is why the homogeneous case is worth treating first — the non-homogeneous system, the independence test, the eigenvector calculation and the kernel are the same reduction asked about differently.