VECTOR SPACES · REFERENCE CARD

The four fundamental subspaces

Two live in $\mathbb{R}^n$ and two in $\mathbb{R}^m$, and within each pair they are orthogonal complements. Every entry gives its dimension and how to compute a basis — all four come out of the same row reduction.

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SUBSPACES

IN THE DOMAIN — $\mathbb{R}^n$ 2

1 Row space

dimension r — the rank

$$\mathrm{Row}(A) = \mathrm{Col}(A^T)$$

Span of the rows. Row-reduce A and take the **nonzero rows of the echelon form** — row operations preserve the row space exactly, which is why the reduced rows still span it. Note the contrast with the column space, where the echelon form must not be used.

2 Null space

dimension $n - r$ — the nullity

$$\mathrm{Null}(A) = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$$

Everything the matrix sends to zero. Reduce to RREF and set each free variable to 1 in turn, the others to 0 — one basis vector per free variable, which is where the count $n - r$ comes from.

IN THE CODOMAIN — $\mathbb{R}^m$ 2

3 Column space

dimension r — the rank

$$\mathrm{Col}(A) = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\}$$

The reachable outputs. Row-reduce to find which columns hold pivots, then take **those columns of the original A** — not of the echelon form. Row operations change the column space, so the reduced columns span something else entirely. This is the single most common error on the page.

4 Left null space

dimension $m - r$

$$\mathrm{Null}(A^T) = \{\mathbf{y} : A^T\mathbf{y} = \mathbf{0}\}$$

The directions no output ever reaches. Row-reduce A^T, or augment as $[A^T \mid I_m]$ to get the combining coefficients as well. Named "left" because $A^T\mathbf{y} = \mathbf{0}$ is the same as $\mathbf{y}^T A = \mathbf{0}^T$.

WHAT TIES THE PAIRS TOGETHER 2

5 Domain split

rank-nullity

$$r + (n - r) = n$$

The row space and null space are orthogonal complements in $\mathbb{R}^n$, so their dimensions sum to n and they meet only at the origin. Every vector in the domain splits uniquely into a part each.

6 Codomain split

the transpose statement

$$r + (m - r) = m$$

The same accounting on the other side: column space and left null space are orthogonal complements in $\mathbb{R}^m$. The shared r is what makes the two splits one theorem rather than two.

One reduction of A yields all four bases. The pairing is what makes the picture close: $\mathrm{Row}(A)$ and $\mathrm{Null}(A)$ split the domain, $\mathrm{Col}(A)$ and $\mathrm{Null}(A^T)$ split the codomain, and the two dimensions they share is the **rank** — which is why row rank equals column rank.