BEFORE YOU READ

Four stages, each extending the last

The definition of a^x was built in four stages. Each stage keeps every exponent the previous one allowed and fills in more of the number line, while preserving the same rules.

$$\mathbb{Z}^+ \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

$\mathbb{R}$ **Irrational exponents** — `2^π ≈ 8.825`

adds the remaining points · continuous extension

$\mathbb{Q}$ **Rational exponents** — `8^(2/3) = 4`

adds every fraction · $a^{m/n} = \sqrt[n]{a^m}$

$\mathbb{Z}$ **Negative exponents** — $2^{-3} = 1/8$

adds zero and the negative integers · $a^0 = 1$, $a^{-n} = 1/a^n$

$\mathbb{Z}^+$ **Natural exponents** — $2^4 = 16$

repeated multiplication

TAKEAWAY

The whole real line

For any base $a > 0$ and any real exponent x, a^x is defined, positive, and obeys every law of exponents. This is the foundation of exponential functions.