

ONLY TWO POSSIBILITIES

Appending $\mathbf{b}$ as an extra column can raise the rank by at most 1. So comparing the two ranks leaves exactly two outcomes.

EXISTENCE

$$A\mathbf{x} = \mathbf{b}$$

split by $\text{rank}([A \mid \mathbf{b}])$ compared with $\text{rank}(A)$



CASE 1

$$\text{rank}([A \mid \mathbf{b}]) = \text{rank}(A)$$

Consistent

$\mathbf{b}$ is a linear combination of the columns; appending it adds no new independent direction



CASE 2

$$\text{rank}([A \mid \mathbf{b}]) = \text{rank}(A) + 1$$

Inconsistent

$\mathbf{b}$ is not in the column space; echelon form shows a row $[0 \cdots 0 \mid d]$ with $d \neq 0$



TAKEAWAY

Existence is a question about $\mathbf{b}$ alone: is it already a linear combination of the columns of A ?