# Properties of the dot product

Four laws hold unconditionally and three familiar habits break. Grouping by verdict makes the split structural — you see how much survives before reading a single entry.

07
ENTRIES

## UNCONDITIONAL 3

### i Commutativity $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

**UNCONDITIONAL**

Inherited directly from commutativity of real-number multiplication — $a_i b_i = b_i a_i$ for every component. Note this is the opposite of **matrix multiplication**, where commutativity fails.

### ii Distributivity $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

**UNCONDITIONAL**

Lets a dot product with a sum be broken apart — used constantly when expanding expressions over **vector addition**.

### iii Scalar factoring $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b})$

**UNCONDITIONAL**

A scalar inside the product can be pulled out. Combined with commutativity this also gives $\mathbf{a} \cdot (c\mathbf{b}) = c(\mathbf{a} \cdot \mathbf{b})$, so it works on either slot.

## CONDITIONAL 1

### iv Positive definiteness $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2 \geq 0$

**CONDITIONAL**

HOLDS WHEN equality iff $\mathbf{a} = \mathbf{0}$

The self-dot product is squared length. This is the axiom that ties the dot product to the **norm** — drop it and lengths stop being meaningful.

## FAILS OR UNDEFINED 3

### v Associativity $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$

**UNDEFINED**

FAILS WHEN always — not well formed

Ungrammatical rather than false. $\mathbf{a} \cdot \mathbf{b}$ is a scalar, and a scalar has no dot product with a vector. The nearest meaningful expression is $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$, which is a scalar multiple.

### vi Cancellation $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} \not\Rightarrow \mathbf{b} = \mathbf{c}$

**FAILS**

FAILS WHEN always, for $\mathbf{a} \neq \mathbf{0}$

The dot product discards whatever is orthogonal to $\mathbf{a}$, so any two vectors agreeing along $\mathbf{a}$ are indistinguishable to it. This is the same collapse that makes **projection** non-invertible.

**WITNESS**

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a = (1, 0), b = (2, 5), c = (2, 9)
a · b = a · c = 2, yet b ≠ c
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### vii Closure $\mathbf{a} \cdot \mathbf{b} \notin \mathbb{R}^n$

**FAILS**

FAILS WHEN always — output is a scalar

Unlike vector addition, the dot product leaves the space. That is what makes it a *form* rather than an operation on vectors, and why it can measure angle and length at all.

The failures all trace to one fact: the dot product returns a scalar, so it leaves the space it operates on. Compare the **cross product**, which stays inside $\mathbb{R}$ and fails associativity for a different reason entirely.