Which matrices diagonalize

One condition decides it and everything else is a shortcut to checking that condition. The multiplicity test is necessary and sufficient; the rows above it are sufficient only, and the rows below are what failure looks like.

6
CASES

SUFFICIENT — SETTLES IT WITHOUT COMPUTING EIGENVECTORS — 3

1 Distinct eigenvalues

sufficient, not necessary

n distinct λ_i

Eigenvectors for distinct eigenvalues are automatically independent, so n of them form a basis. The converse fails and the failure is common — the identity matrix has one eigenvalue repeated n times and diagonalizes trivially.

2 Real symmetric

always, and orthogonally

$A = A^{\mathrm{T}} \Rightarrow A = QDQ^{\mathrm{T}}$

The **spectral theorem**. Symmetry guarantees not only diagonalizability but an orthonormal eigenvector basis, so P^{-1} is P^{T} and nothing needs inverting. No computation required to know it applies.

3 Already diagonal

A diagonal, including I and cI

$P = I$

Degenerate but worth stating, because it is the counterexample to distinctness being necessary. A scalar matrix cI has one eigenvalue of multiplicity n and every vector is an eigenvector.

THE DEFINITIVE TEST — 1

4 Multiplicity comparison

necessary **and** sufficient

$m_g(\lambda_i) = m_a(\lambda_i)$ for every i

Geometric multiplicity is $\dim \mathrm{Null}(A - \lambda I)$, algebraic is the root multiplicity in the characteristic polynomial. Geometric never exceeds algebraic, so the test is whether any eigenvalue falls short — and one shortfall is enough to fail.

FAILS — TOO FEW EIGENVECTORS — 2

5 Defective matrix

some eigenvalue is short

$m_g(\lambda) < m_a(\lambda)$

Not enough independent eigenvectors to form a basis, so no P exists. The Jordan form is the substitute — as close to diagonal as the matrix permits, with ones on the superdiagonal recording exactly how many eigenvectors are missing.

6 Nonzero nilpotent

never diagonalizable

$A^k = O, A \neq O$

Every eigenvalue is zero, so a diagonalization would force $D = O$ and hence $A = PDP^{-1} = O$. The zero matrix is the only diagonalizable nilpotent — which makes shears and other nilpotent-plus-identity matrices the standard examples of defectiveness.

Everything reduces to counting eigenvectors. A matrix diagonalizes exactly when it has n independent ones, and the multiplicity comparison is that count performed eigenvalue by eigenvalue. The shortcuts are worth knowing because they settle the common cases without computing a single eigenvector.