A SHARP DIVIDING LINE

The value $\det(A) = 0$ splits square matrices in two. The determinant answers the single most important structural question about a square matrix in a single number.

SQUARE MATRIX

$\det(A)$, A is $n \times n$

split by **whether** $\det(A) = 0$

0

CASE 1

$\det(A) = 0$

Singular

the columns are linearly dependent, $Ax = \mathbf{0}$ has nontrivial solutions, the rank is less than n, and at least one dimension collapses

> More

✓

CASE 2

$\det(A) \neq 0$

Nonsingular (invertible)

the columns form a basis, $Ax = \mathbf{b}$ has exactly one solution for every $\mathbf{b}$, and A^{-1} exists

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✓ **TAKEAWAY**

A nonsingular matrix stretches or compresses volumes by $|\det(A)|$ without collapsing any dimension; a singular one flattens regions of positive volume down to zero volume.