DETERMINANTS · SUMMARY

What each operation does to the determinant

Eight properties, each linked to the section that establishes it. The three row operations at the top are the ones that make computation by reduction possible — everything below follows from them.

08
PROPERTIES

UNCONDITIONAL 2

iv Row addition $R_i + cR_k \to R_i \;\Rightarrow\; \det(B) = \det(A)$

UNCONDITIONAL

REQUIRES $i \neq k$

The determinant is untouched. This is the operation **Gaussian elimination** leans on hardest — it can be applied freely without bookkeeping, which is what makes reduction a practical route to the determinant.

v Transpose invariance $\det(A^T) = \det(A)$

UNCONDITIONAL

REQUIRES A square

Every statement about rows has a column counterpart, and neither is primary. Cofactor expansion may be taken along any row or any column for the same reason.

CONDITIONAL 5

i Row swap $R_i \leftrightarrow R_k \;\Rightarrow\; \det(B) = -\det(A)$

SIGN FLIPS

HOLDS WHEN any two distinct rows exchanged

The only operation that changes the sign. An even number of swaps restores it, which is why the sign of a permutation is well defined and why the determinant can be built from permutations at all. Swapping a row with itself changes nothing, consistent with the rule.

ii Row scaling $kR_i \to R_i \;\Rightarrow\; \det(B) = k\det(A)$

CONDITIONAL

HOLDS WHEN one row scaled by k

COMMON ERROR applying this to the whole matrix — see the next entry

The determinant is linear in each row separately. Scaling a single row by k scales the determinant by k; scaling by $k = 0$ produces a zero row and a zero determinant, which is the same fact as a dependent row set.

iii Scaling the whole matrix $\det(kA) = k^n \det(A)$

CONDITIONAL

HOLDS WHEN A is $n \times n$

COMMON ERROR writing $\det(kA) = k\det(A)$

Scaling every entry means scaling all n rows, so the factor applies n times. This is the single most common determinant error, and it comes from reading the row-scaling rule one line too broadly.

WITNESS

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A = I₂, det(A) = 1
det(2A) = det([[2,0],[0,2]]) = 4 = 2² · 1, not 2
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vi Multiplicative property $\det(AB) = \det(A)\det(B)$

CONDITIONAL

HOLDS WHEN A and B both $n \times n$

The determinant is a homomorphism from matrix multiplication to real multiplication. Geometrically the volume scaling factors compose, which is the whole content of the identity. Note there is deliberately no corresponding rule for sums.

vii Determinant of the inverse $\det(A^{-1}) = \dfrac{1}{\det(A)}$

CONDITIONAL

HOLDS WHEN $\det(A) \neq 0$

Falls straight out of the multiplicative property applied to $AA^{-1} = I$. It also proves a singular matrix has no **inverse** — no real number multiplied by zero gives one.

FAILS OR UNDEFINED 1

viii Sum of matrices $\det(A + B) \neq \det(A) + \det(B)$

FAILS

FAILS WHEN in general — no rule exists

Listed because its absence is otherwise inferred from silence. The determinant is linear in each row separately, which is a much weaker statement than being linear in the matrix, and the two are easy to conflate.

WITNESS

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A = [[1,0],[0,0]], B = [[0,0],[0,1]]
det(A) + det(B) = 0, but det(A + B) = det(I₂) = 1
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The pattern worth carrying away: operations that preserve the row space leave the determinant alone or scale it predictably, and only a swap changes its sign. That is why **row reduction** can compute a determinant at all — every step's effect is known in advance.