DETERMINANTS · GEOMETRY

What the determinant measures, by dimension

The magnitude is a volume, the sign is an orientation, and zero is a collapse. Only the word for "volume" changes as the dimension rises — which is the argument for calling the general case a volume at all.

5
DIMENSIONS

WHERE THE PICTURE IS AVAILABLE — 3

1

One dimension

`length scaling; sign = direction`

$\det[a] = a$

A segment on a line. The determinant is the number itself, and its sign says whether the direction is preserved or reversed — the degenerate case that makes the general statement believable rather than surprising.

2

Two dimensions

`sign: counterclockwise +, clockwise −`

$|\det A| =$ area of the parallelogram

The columns of A span a parallelogram, and the determinant is its signed area. Zero means the columns are parallel and the parallelogram has collapsed to a line — which is the same statement as the columns being **linearly dependent**.

3

Three dimensions

`sign: right-handed +, left-handed −`

$|\det A| =$ volume of the parallelepiped

Three columns span a parallelepiped. The sign distinguishes a right-handed frame from a left-handed one — the same distinction the **cross product** encodes, and the reason $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is a determinant.

WHERE IT IS NOT — 2

4

Four dimensions and above

`orientation preserved + or reversed −`

$|\det A| =$ volume of the image of the unit cube

No picture, and none needed — the algebra is unchanged. This is where the determinant stops being a description of something already visible and becomes the definition of n-dimensional content, which is what the **change of variables formula** then relies on.

5

Determinant zero

`image lies in a proper subspace`

$\det A = 0$

The collapse case in every dimension: the image has zero content because it fits inside something lower-dimensional. Note the determinant reports *that* a collapse happened but not by how much — a matrix flattening $\mathbb{R}^3$ to a plane and one flattening it to a point both give zero. That is what **rank** is for.

Read down the column and the pattern is one definition stated four ways: $|\det|$ is the factor by which the unit cube's content is scaled, the sign says whether the image is reflected, and $\det = 0$ says the image has no content because it fits inside a smaller subspace. Length, area and volume are not analogies here — they are the $n = 1, 2, 3$ cases of the same quantity.