Six places a determinant does the work

Three of these produce a value and three produce a verdict. The split matters because the first group is almost never the fastest route to that value — while the second group has no competitor at all.

6

APPLICATIONS

PRODUCING A VALUE — CORRECT, RARELY FASTEST 3

1 Cramer's rule

$\det(A) \neq 0$; A_i has column i replaced by $\mathbf{b}$

$$x_i = \frac{\det(A_i)}{\det(A)}$$

Each component as a ratio of determinants. Costs $n+1$ determinants against one **elimination**, so it is a formula rather than a method — valuable because it exhibits the solution as an explicit function of the entries, which is what sensitivity analysis needs.

2 Adjugate inverse

$\det(A) \neq 0$

$$A^{-1} = \frac{1}{\det A}\operatorname{adj}(A)$$

Every entry of the inverse in closed form. Practical at 2×2 and defensible at 3×3; beyond that the cofactor count makes it unusable numerically. Its real work is theoretical — it proves the inverse entries are rational functions of A.

3 Structured determinants

the matrix has exploitable structure

Vandermonde: $\prod_{i<j}(x_j - x_i)$

Closed forms that skip the computation entirely. The Vandermonde value is nonzero exactly when the nodes are distinct, which is the statement that polynomial interpolation through distinct points has a unique solution.

PRODUCING A VERDICT — NOTHING ELSE DOES IT AS WELL 3

4 Characteristic polynomial

roots are the **eigenvalues**

$$p(\lambda) = \det(A - \lambda I)$$

The determinant is what converts "this homogeneous system has a nontrivial solution" into a polynomial equation in λ. Without it there is no equation to solve, which is why eigenvalue theory begins here rather than with eigenvectors.

5 Cross product

$\mathbb{R}^3$ only

$$\mathbf{a} \times \mathbf{b} = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ & \mathbf{a} & \\ & \mathbf{b} & \end{bmatrix}$$

A mnemonic that is also an explanation. Expanding along the first row reproduces the component formula, and the determinant being zero when two rows match is exactly why $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ — see **cross product**.

6 Wronskian

$W \neq 0$ somewhere ⇒ independent

$$W(x) = \det[f_j^{(i)}(x)]$$

One-directional, and the direction matters. A nonzero value at a single point proves independence; $W \equiv 0$ proves nothing in general — x^2 and $x|x|$ are independent with W identically zero. The converse holds only for solutions of one linear ODE.

Where a determinant computes something, elimination usually computes it faster; where a determinant decides something, nothing else decides it as cleanly. That is the honest summary of the determinant's role: a poor algorithm and an excellent criterion.