VECTORS · CROSS PRODUCT

# Properties of the cross product

Linear in each slot and distributive, like the dot product — but anti-commutative rather than commutative, and not associative at all. The two failures are the ones worth holding onto.

06

ENTRIES

UNCONDITIONAL 2

### ii Distributivity $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

UNCONDITIONAL

REQUIRES spreads over sums on either side

Combined with scalar factoring this makes the cross product bilinear — linear in each slot separately, which is what lets it be computed by the **determinant formula**.

### iii Scalar factoring $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$

UNCONDITIONAL

REQUIRES any scalar $c$

Scalars migrate freely between slots and out front. A negative $c$ reverses the output, consistent with anti-commutativity.

CONDITIONAL 2

### i Anti-commutativity $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$

ORDER FLIPS SIGN

HOLDS WHEN always — the sign is the rule, not an exception

Swapping the inputs reverses the output vector. The magnitude is unchanged, so the parallelogram area is the same; only the perpendicular direction flips, which is the right-hand rule reading itself backwards.

### iv Self-cross product $\mathbf{a} \times \mathbf{a} = \mathbf{0}$

ALWAYS ZERO

HOLDS WHEN for every $\mathbf{a}$, including $\mathbf{0}$

Forced by anti-commutativity: $\mathbf{a} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{a})$, and the only vector equal to its own negative is $\mathbf{0}$. Geometrically a vector spans no parallelogram with itself, so there is no area and no unique perpendicular.

FAILS OR UNDEFINED 2

### v Associativity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$

FAILS

FAILS WHEN in general — parenthesisation changes the result

The triple product expansion $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$ is the workaround: it rewrites a nested cross product without one.

> **WITNESS**
> ```
> a = i, b = i, c = j
> a × (b × c) = i × (i × j) = i × k = -j
> (a × b) × c = (i × i) × j = 0 × j = 0
> ```

### vi Cancellation $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c} \not\Rightarrow \mathbf{b} = \mathbf{c}$

FAILS

FAILS WHEN always, for $\mathbf{a} \neq \mathbf{0}$

The cross product discards whatever is parallel to $\mathbf{a}$ — the mirror image of the **dot product**, which discards whatever is perpendicular. Between them nothing is lost, which is why the two together determine a vector completely.

> **WITNESS**
> ```
> a = (1, 0, 0), b = (0, 1, 0), c = (1, 1, 0)
> a × b = a × c = (0, 0, 1), yet b ≠ c
> ```

Both failures come from the same source: the output is a vector perpendicular to its inputs, so it depends on their order and on which pair you multiply first. The **dot product** returns a scalar and fails in the opposite direction — it commutes freely but leaves $\mathbb{R}$ entirely.